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Hyperbolic Systems of Conservation Laws in Several Space Variables

P. D. Lax

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CONTENTS

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Introduction

In Section 1 we show how to associate to each analytic function f of one variable a complex conservation law. When written in terms of real variables, this equation turns into a symmetric hyperbolic system of two first order conservation laws in time and two space variables for two unknowns. The hope is that the theory of analytic functions can be exploited to study the properties of the solutions of these model equations.

These systems are strictcly hyperbolic where $f' \neq 0$, and nonlinear where $f'' \neq 0$. There is, however, in every state at least one direction of propagation which is not genuinely nonlinear.

In Section 2 we show that every 2×2 quasilinear first order hyperbolic system fails to be genuinely nonlinear in some direction of propagation.

The notions of the first section are asily extended to analytic functions of several variables.

This paper is dedicated to Sigeru Mizohata, in recognition of his deep contributions to our understanding of the theory of partial differential equations.

1. Symmetric Hyperbolic Systems of Conservation Laws.

$f = f(U)$ denotes an analytic function of a single complex variable U. We impose on the complex valued function $U(z,t)$ of the complex variable z and the real variable t the following nonlinear partial differential equation:

$$\partial_t \overline{U} + \partial_z f(U) = 0 . \tag{1.1}$$

Here, as customary, ∂_z denotes

$$\partial_z = \frac{1}{2}(\partial_x - i\ \partial_y), \qquad z = x + iy \tag{1.2}$$

and the bar denotes the complex conjugate. We can express this equation in terms of the real and imaginary parts of U and f_U:

$$U = u + iv\ , \quad \frac{1}{2} f_U = a + ib\ . \tag{1.3}$$

Setting these into (1.1) gives

(1.4) $$u_t - iv_t + (a+ib)(u_x+iv_x-iu_y+v_y) = 0 \ ,$$

The matrix form of this equation is

(1.4)´ $$\binom{u}{v}_t + A\binom{u}{v}_x + B\binom{u}{v}_y = 0$$

where

(1.5) $$A = \begin{vmatrix} a & -b \\ -b & -a \end{vmatrix} \ , \quad B = \begin{vmatrix} b & a \\ a & -b \end{vmatrix}$$

Since A and B are symmetric matrices, (1.4)´ is a <u>symmetric hyperbolic system</u>.

The characteristics of this sytem are the eigenvalues of $\xi A + \eta B$; these are

(1.6) $$\pm \sqrt{a^2+b^2} \quad \text{for} \quad \xi^2+\eta^2 = 1 \ .$$

Thus the speed with which signals propagate from a point is $\sqrt{a^2+b^2}$ in all directions.

It was observed in [3], see also [5] and [2], that a symmetric system of conservation laws satisfies an additional conservation law. This can be derived from the complex form (1.1) of the equation by multiplying it by U:

$$UU\overline{U}_t + Uf_U U_z = 0 \; . \tag{1.7}$$

Denote by h the analytic function, determined up to a constant, whose derivative is f:

$$h_U = f \tag{1.8}$$

Then

$$Uf_U = (Uf - h)_U = k_U \; . \tag{1.8)´}$$

Equation (1.7) can be rewritten as

$$\overline{U}U_t + k_z = 0 \; . \tag{1.8)"}$$

Add the complex conjugate of (1.8) to (1.8"):

$$\partial_t |U|^2 + k_z + \overline{k_{\bar{y}}} = 0 \; . \tag{1.9}$$

$|U|^2$ is a convex function of U; as explained in [5], it plays the role of an entropy for equation (1.1), in the following sense: if U is a strong limit as $\varepsilon \to 0$ of solutions $U^{(\varepsilon)}$ of the viscous equations

$$\overline{U}_t^{(\varepsilon)} + f(U^{(\varepsilon)})_z = \varepsilon\Delta\overline{U}^{(\varepsilon)}. \tag{1.10}$$

then U satisfies the entropy inequality

$$\partial_t |U|^2 + k_z + \overline{k_{\bar{z}}} \le 0 \ . \tag{1.11}$$

At a point of discontinuity of U, called a shock, the strict inequality holds in (1.11).

We note that every <u>antianalytic</u> function of U is a time-independent solutiion of (1.1).

We show now how to associate to each analytic function h of several complex variables a symmetric system of conservation laws.

Let $h(U^1,\ldots,U^n)$ be an analytic function; denote its partial complex derivatives by

$$f^j = h_{U^j} \ . \tag{1.12}$$

The system of n complex conservation laws

$$\overline{U}_t^j + f^j_z = 0 \ , \quad j = 1,\ldots,n, \tag{1.13}$$

when written in terms of the real and imaginary parts of U^j, is a symmetric system of conservation laws; this is easy to verify

directly, but it is still easier to verify an equivalent fact, that $\Sigma\ |U^j|^2$ satisfies an additional conservation law. To see this multiply the j^{th} equation (1.13) by U^j and sum over j:

$$(1.14) \qquad \Sigma\ U^j\ \overline{U^j_t} + \Sigma\ U^j f^j_z = 0$$

Using (1.12) we see that the second term can be written as

$$(1.15) \qquad k_z = (\Sigma\ U^j f^j - h)_z\ .$$

Taking the real part of (1.14) and using (1.15) gives

$$(1.16) \qquad \partial_t\ \Sigma\ |U^j|^2 + k_z + \overline{k_{\bar{z}}} = 0$$

Equation (1.16) holds for all smooth solutions of (1.13); for discontinuous solutions the corresponding inequality

$$(1.16) \qquad \partial_t\ \Sigma\ |U^j|^2 + k_z + \overline{k_{\bar{z}}} = 0$$

serves as entropy condition.

2. Linear Degenerations of Quasilinear Equations

A first order system in t and one space variable,

$$(2.1) \qquad u_t + Au_x = 0\ ,$$

u a vector function with values in $\mathbb{R}^n$, A an $n\times n$ real matrix, is called strictly hyperbolic if the matrix A has n real and distinct eigenvalues $\lambda_1,\ldots,\lambda_n$. When (2.1) is quasilinear, i.e. when A depends on u, so do the eigenvalues $\lambda_1,\ldots,\lambda_n$ and the corresponding right eigenvectors $r_1,\ldots,r_n$:

$$Ar = \lambda r . \tag{2.2}$$

The system (2.1) is called genuinely nonlinear if for each eigenvalue $\lambda_i = \lambda$, the gradient of λ satisfies for all u the inequality

$$\lambda_u \cdot r \neq 0 , \tag{2.3}$$

where r is the corresponding right eigenvector. Note that (2.3) is independent of the normalization r.

For $n = 1$, (2.1) is a scalar equation:

$$u_t + a\, u_x = 0 ,$$

$a = a(u) = A(u)$. In this case there is a single eigenvalue, a itself, and condition (2.3) requires that

$$a_u \neq 0 , \tag{2.3$'$}$$

clearly the condition of genuine nonlinearity.

The significance of genuine nonlinearity for the theory of hyperbolic equations and the formation of singularities is explained in any treatise on the subject such as [6] or [8].

We turn now to first order systems in two space variables:

$$u_t + Au_x + Bu_y = 0 , \tag{2.4}$$

u an $\mathbb{R}^n$ valued function of x,y,t, A and B real $n \times n$ matrices.

Equation (2.4) has solutions u that propagate in the direction (ξ,η), i.e. are of the form

$$u(x,y,t) = w(\xi x + \eta y, t) \tag{2.5}$$

Such a u satisfies (2.4) if $w(s,t)$ satisfies

$$w_t + C w_s = 0 , \tag{2.6}$$

where

$$C = \xi A + \eta B . \tag{2.7}$$

Equations (2.4) is called <u>strictly hyperbolic</u> if equation (2.6) characterizing those solutions that propagate in the direction (ξ,η) is strictly hyperbolic for every choice of the

direction. That means that the matrix C has to have real and distinct eigenvalues for all real choices of ξ, η, $\xi^2+\eta^2 > 0$.

Suppose equation (2.4) is quasilinear, i.e., that A and B are functions of u. We call (2.4) genuinely nonlinear if the equations (2.6) for unidirectional waves are genuinely nonlinear for every choice of the direction.

Take a scalar equation

$$u_t + au_x + bu_y = 0 \ ,$$

a,b functions of u; equation (2.6) is

$$w_t + c\, w_s = 0 \ ,$$
$$c = \xi a + \eta b \ .$$

Condition (2.3)′ becomes

$$\xi a_u + \eta b_u \neq 0 \ ,$$

clearly violated by suitable choice of (ξ,η). Thus a scalar quasilinear equation in two space variables is never genuinely nonlinear in all directions.

We turn now to the 2 × 2 system (1.4)′. We write

$$a + ib = qe^{i\phi}$$

i.e.

$$a = q \cos \phi , \quad b = q \sin \phi ,$$

so that the matrices A, B in (1.5) can be written as

$$A = q \begin{vmatrix} \cos \phi, & -\sin \phi \\ -\sin \phi, & -\cos \phi \end{vmatrix} , \quad B = q \begin{vmatrix} \sin \phi, & \cos \phi \\ \cos \phi, & -\sin \phi \end{vmatrix}$$

Setting

$$\xi = \cos \theta , \quad \eta = \sin \theta$$

we get for C given by (2.7)

$$D = q \begin{vmatrix} \cos (\theta-\phi) & \sin(\theta-\phi) \\ \sin (\theta-\phi) & -\cos(\theta-\phi) \end{vmatrix}$$

The eigenvalues of C are $\pm$ q; a brief calculation shows that the right eigenvectors are

(2.8) $$r_+ = \begin{vmatrix} \cos \frac{\theta-\phi}{2} \\ \sin \frac{\theta-\phi}{2} \end{vmatrix} , \quad r_- = \begin{vmatrix} \sin \frac{\theta-\phi}{2} \\ -\cos \frac{\theta-\phi}{2} \end{vmatrix}$$

Condition (2.3) in this case is

$$q_u \cdot r_{\pm} \neq 0 \ . \tag{2.9}$$

We claim that this cannot hold for all θ. For q, and therefore q_u is independent of θ; on the other hand formulas (2.8) show that as θ goes from 0 to 2π, the vectors r_+ and r_- turn over an angle of π; therefore there are values $\theta_{\pm}$ when $r_{\pm}(\theta_{\pm})$ are orthogonal to g_u . For these values of θ condition (2.9) is violated.

Andy Majda raised the question if there genuinely are nonlinear 2×2 systems in two space dimensions. It turns out that indeed there aren't any:

<u>Theorem</u>. Every real, strictly hyperbolic quasilinear system (2.4) for two unknowns of two space variables is linearly degenerate in some direction.

<u>Proof</u>: Define

$$C(\theta) = \cos\theta\, A + \sin\theta\, B \ . \tag{2.10}$$

Denote the eigenvalues of $C(\theta)$ by $\lambda_{\pm}(\theta)$, labeled so that

$$\lambda_-(\theta) < \lambda_+(\theta) \ ; \tag{2.11}$$

the corresponding right eigenvectors are denoted by $r_{+}(\theta)$. We choose them to be real and of unit length:

$$(2.12) \qquad |r_{\pm}(\theta)| = 1 ,$$

in any old norm. This still leaves an arbitrary factor ± 1, which we fix arbitrarily at $\theta = 0$, and for all other θ in $0 \leq \theta \leq 2\pi$ by requiring $r_{\pm}(\theta)$ to vary continuously with θ.

The matrix $C(\theta)$ defined by (2.10) satisfies

$$(2.13) \qquad C(\theta + \pi) = - C(\theta) .$$

It follows from (2.13) and (2.11) that

$$\lambda_{+}(\theta + \pi) = - \lambda_{-}(\theta)$$
$$\lambda_{-}(\theta + \pi) = - \lambda_{+}(\theta) .$$

It follows from this and (2.12) that

$$(2.14) \qquad \begin{aligned} r_{+}(\theta+\pi) &= \sigma_{+} r_{-}(\theta) \\ r_{-}(\theta+\pi) &= \sigma_{-} r_{+}(\theta) \end{aligned}$$

where σ_{+} and σ_{-} are ± 1. Since $r_{\pm}(\theta)$ were chosen to be continuous functions of θ, $\sigma_{\pm}$ also are continuous functions; since their value is ± 1, they are constant.

Since $r_{\pm}(\theta)$ vary continuously, the orientation of the ordered base

$$\{r_-(\theta),\ r_+(\theta)\}$$

does not change; in particular

$$\{r_-(0), r_+(0)\} \quad \text{and} \quad \{r_-(\pi), r_+(\pi)\}$$

have the same orientation. By (2.14)

$$\{r_-(0), r_+(0)\} \quad \text{and} \quad \{\sigma_- r_+(0), \sigma_+ r_-(0)\}$$

have the same orientation; this proves that

$$\sigma_+\sigma_- = -1 . \tag{2.15}$$

Setting $\theta = 0$ and $\theta = \pi$ we obtain

$$r_+(2\pi) = \sigma_+ r_-(\pi) = \sigma_+\sigma_- r_+(0) ;$$

so by (2.15)

$$r_+(2\pi) = -\, r_+(0) \tag{2.16$_+$}$$

and similarly

$$(2.16) \qquad r_-(2\pi) = - r_-(0) .$$

The eigenvalues $\lambda_\pm(u,\theta)$ are periodic functions of θ with period 2π; therefore so is their gradient; this and (2.16) shows that

$$(2.17) \qquad \lambda_u(2\pi) \cdot r(2\pi) = - \lambda_u(0) \cdot r(0)$$

for both eigenvalue-vector pairs. Since

$$(2.18) \qquad \lambda_u(\theta) \cdot r(\theta)$$

varies continuously with θ, it follows from (2.17) that (2.18) vanishes for some θ. By (2.3) the system (2.4) fails to be genuinely nonlinear in the direction $(\cos\theta, \sin\theta)$. ∎

The argument presented above shows this

<u>Corollary</u>. Let (2.4) be a $2n \times 2n$ real, quasilinear, strictly hyperbolic system in two space variables, n odd. Then (2.4) fails to be genuinely nonlinear in some direction.

The topological argument used in the above proof is taken from [7], where it is used to prove that there are <u>no</u> strictly hyperbolic systems of $2n$ equations for $2n$ unknowns in <u>three</u> space

variables for n odd. The interesting question of the existence of $m \times m$ strictly hyperbolic systems in three variables is handled for arbitrary m in the interesting paper [1].

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